Name ______________________________

Probability and the Birthday Paradox

Introduction

In his *Beauty and Mathematics* Lecture on DVD, Dr. Wang asserts that in a group of 23 people, there is a greater than 1 in 2 chance that two people will share the same birthday. This surprising fact is known as the "Birthday Paradox" and can be explained using basic principles of probability.

A. Basic Terminology and Principles of Probability

Probability is the mathematics of chance. Probability theory as we know it today has its roots in letters between French mathematicians Blaise Pascal and Pierre Fermat in the mid-1600s.

We list some of the basic terms of probability and then use them in the examples that follow.

Random: having no definite pattern; cannot be predicted with certainty; without any conscious method or decision.

For example, the results of a coin toss or the rolling of a die are random. We do not know with certainty what the result of a coin toss or the rolling of the die will be.

Outcome: any result of a random experiment.

For example, rolling a die yields 6 possible outcomes: 1, 2, 3, 4, 5, or 6; flipping a coin yields two possible outcomes: heads or tails.

Unless otherwise stated, it is generally assumed that the individual outcomes of a random experiment are equally likely to occur over the long run.

For example, in flipping a coin, we assume the coin is "fair," meaning that it will come up heads as many times it comes up tails over the long run.

Sample Space: the set of all outcomes of an experiment.

For example, the sample space of outcomes of rolling a die are the numbers 1 through 6.

Event: a set of particular outcomes

For example, the outcome of rolling an odd number on a die could be considered an event.

If an event is just a single outcome, it is referred to as a **simple event.** For example, rolling the number 5 on a die is called a simple event.

Mutually Exclusive Events: events that cannot occur simultaneously. We say that two individual outcomes are mutually exclusive events. For example, rolling a 4 and rolling a 6 on a die are considered two mutually exclusive events since they cannot occur simultaneously.

The Probability of an Event Occurring: The probability of an event occurring, denoted by P(E), is equal to the number of outcomes in which the event E can occur divided by the total number of possible outcomes.

$$P(E) = \frac{\textit{number of ways E can occur}}{\textit{total number of possible outcomes}}$$

Note that probabilites are expressed as fractions or as numbers that are equivalent to the fractions. We see in the example below that the answer can be expressed in three forms: as a fraction, as a decimal, and as a percent.

Example 1:
A coin is flipped. What is the sample space for this experiment? What is the probability of the simple event that the coin will come up heads?

Solution:
The sample space is the set of all possible outcomes. In the case of flipping a coin, the sample space is: {heads, tails}. Since the event of flipping a heads is a single outcome, the event is called a simple event. We apply the formula for the probability of a simple event occuring:

$$P(\text{heads}) = \frac{1}{2}$$

Note that 1 is the number of different ways the coin can achieve heads and 2 is the total number of possible outcomes. We give this problem for illustrative purposes as you should already know that the probability of flipping a coin and having it come up heads is $\frac{1}{2}$.

The answer to this question can also be expressed as: .5 or 50%.

Example 2:
A die is rolled. What is the sample space of this experiment? What is the probability of the simple event of having it come up a 3?

Solution:
The set of possible outcomes are: 1, 2, 3, 4, 5, and 6.

The probability of rolling a three is $\frac{1}{6}$ where 1 is the total number of different ways the event (coming up a three) can occur and 6 is the total number of outcomes.

Example 3:
What is the probability of the simple event of drawing a spade from a standard deck of cards (one with 52 cards where each suit (spades, hearts, diamonds, clubs) has 13 cards: 2 through 10 + Jack, Queen, King, and Ace).

Solution:
There are 13 ways that the simple event (drawing a spade) can occur and a total number of 52 possible outcomes. Therefore:

$$P(\text{drawing a spade}) = \frac{13}{52}$$

If we simplify the answer and express it in equivalent ways, we get $\frac{1}{4}$, which can also be expressed as .25 or 25%.

B. Probability of the Complement of an Event and of Events in Sequence.

We present below some basic definitions and principles.

Complement of an Event: The complement of an event, E, are those outcomes that are not included in the set of outcomes that are contained in E. (For example, the complement of the event of rolling a 2 or a 3 with a die is the event of rolling a 1, 4, 5, or 6.)

Take careful note of the following important fact:

> The probability of the complement of an event E occurring is equal to 1 – P(E).

Example 4:
Using the formula for the probability of the complement of an event, determine the probability of drawing a card at random from a normal deck of cards (one that has no jokers) that is not a spade.

Solution:
Remembering that a normal deck of cards without jokers contains 52 cards 13 of which are spades, we see that

$$\text{P(drawing a spade)} = \frac{13}{52} = \frac{1}{4}$$

This means that there is a one in four chance of drawing a spade from a deck of cards.

The question, however, asks for the probability of drawing a card that is not a spade – in other words, the complement of the event of drawing a spade. Remembering that the probability of the complement of an event E occurring is 1 - P(E), we see that

$$\text{P(not drawing a spade)} = 1 - \frac{1}{4} = \frac{3}{4}$$

Another important fact about probability concerns the calculation of the probability of two events in sequence:

> The probability of an event E occurring and then an event, F, denoted P(E and F), occuring is P(E) • P(F).

Example 5:
What is the probability of tossing a coin twice and having it come up heads both times?

Solution:
We apply the formula for the probability of events in sequence P(E and F) = P(E) • P(F). Let E stand for the event of tossing a coin once and having it come up heads, and let F stand for the event of tossing a coin a second time and having it come up heads.

Applying the formula, we see that:

$$P(E) = \frac{1}{2} \quad \text{and} \quad P(F) = \frac{1}{2}$$

Therefore:

$$P(E \text{ and } F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Even if a coin were tossed 100 times and came up heads all 100 times, the probability of tossing heads on the 101st toss is still $\frac{1}{2}$.

In Example 5, the two events are said to be **independent.** Two events are said to be independent if the outcome of one event does not in any way affect the outcome of the second event. Whether the first coin toss came up heads or tails does not affect the whether the second coin toss will come up heads or tails.

In the next example, we will compute the probability of two events that are **dependent.** Two events are said to be dependent if the outcome of one event does affect the outcome of the other event.

Example 6:
What is the probability of the event of drawing two cards from a regular deck of cards (one with no jokers) and getting two cards that are spades?

Solution:
The two events listed here are dependent events as the outcome of the second event is affected by the outcome of the first event.

If the first card drawn is not a spade, then the probability of drawing a spade for the second card is $\frac{13}{51}$, as there are 13 spades still remaining in a deck which now has only 51 cards . If the first card is a spade, then the probability of drawing a spade for the second card is $\frac{12}{51}$.

We let E designate the event of drawing the first card and that card being a spade. We let F designate the event of drawing the second card and that card being a spade (assuming the first card drawn was a spade).

Therefore,

$$P(E \text{ and } F) = \frac{13}{52} \cdot \frac{12}{51} = \frac{1}{17}$$

C.The "Birthday Paradox" Problem

In the worksheet accompanying this handout, you will work through all the steps necessary to calculate the probability that two people in a room have the same birthday for different numbers of people.

In doing this calculation, you will see that in a group of just 23 people, there is a greater than 50% chance that two people do have the same birthday.

The calculation involves applying several of the definitions and facts in Sections A and B. The "trick" to doing the problem is to calculate the probability that two people in a group share the same birthday.

We assume that there are 365 days in a year. We also assume that it is equally probable that a person will have a birthday on a particular day of the year. That is,

$$P(\text{a person will have a birthday on a particular day of the year}) = \frac{1}{365}$$