Beauty and Mathematics
Expedition 003, Worksheet Solutions
Probability and the Birthday Paradox

Name ______________________________

A. Basic Terminology and Principles of Probability

1. heads, tails
2. 1, 2, 3, 4, 5, 6
3. outcomes: boy, girl

   $P(\text{boy}) = \frac{1}{2} = 50\%$

4. There are 26 letters in the alphabet, only one of which is "r."

   $P(\text{r}) = \frac{1}{26}$

5. outcomes: 1, 2, 3, 4, 5, 6

   $P(4) = \frac{1}{6} = 0.1\overline{6}$

6. There are 13 heart cards in a normal deck of 52 cards.

   $P(\text{heart}) = \frac{13}{52} = \frac{1}{4} = 25\%$

7. In a normal deck of cards, there are four 6s (6 of spades, 6 of hearts, 6 of diamonds, 6 of clubs) and 52 total cards.

   $P(6) = \frac{4}{52} = \frac{1}{13}$

8. There are two white socks and eight total socks in the box.

   $P(\text{white sock}) = \frac{2}{8} = \frac{1}{4}$

9. There are two black socks and two blue socks in the box, and there are eight socks in all.

   $P(\text{black or blue sock}) = \frac{4}{8} = \frac{1}{2}$

10. The question can be reworded to say "What is the probability of rolling a 1, 2, 3, 4, or 5?"
    outcomes: 1, 2, 3, 4, 5, 6
    There are six outcomes, and five of them are not 6 (or said another way, five are 1, 2, 3, 4, or 5).

    $P(\text{not } 6) = P(1, 2, 3, 4, \text{or } 5) = \frac{5}{6}$

B. Probability of the Complement of an Event and of Events in Sequence

1. a. In a deck of 52 cards, only one card is the Queen of Hearts.

   $P(\text{Queen of Hearts}) = \frac{1}{52}$

b. P(not Queen of Hearts) $= 1 - \frac{1}{52} = \frac{51}{52}$

c. $\frac{1}{52} + \frac{51}{52} = 1$

2. a. Each toss is an independent event.

P(heads on first toss) $= \frac{1}{2}$

P(heads on second toss) $= \frac{1}{2}$

P(heads on third toss) $= \frac{1}{2}$

These are events in sequence, so the probability of all three events occurring is the product of the individual probabilities.

P(heads on all three tosses) $= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

b. This is the complement of the event in 2a.

P(not heads all three times) $= 1 - \frac{1}{8} = \frac{7}{8}$

3. There are four Queens (Queen of Spades, Queen of Hearts, Queen of Diamonds, Queen of Clubs) in a standard deck of 52 cards.

P(Queen on first draw) $= \frac{4}{52} = \frac{1}{13}$

After drawing one card, only 51 cards remain in the deck. If the first card drawn was a Queen then only three Queens remain; otherwise there are still four.

P(Queen on second draw if first draw was Queen) $= \frac{3}{51} = \frac{1}{17}$

These are events in sequence.

P(Queens on both draws) $= \frac{1}{13} \cdot \frac{1}{17} = \frac{1}{221}$

4. P(heads) $= \frac{1}{2}$

In a normal deck of cards, the 13 clubs and 13 spades are black while the 13 hearts and 13 diamonds are red.

P(black card) $= \frac{13 + 13}{52} = \frac{26}{52} = \frac{1}{2}$

These are events in sequence.

P(flipping heads and drawing black card) $= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

5. P(July the 4th) $= \frac{1}{365}$

P(not July the 4th) $= 1 - \frac{1}{365} = \frac{364}{365}$

C. The "Birthday Paradox" Problem

1. a. Person A puts an X on one of the 365 boxes.

   P(Person B puts X on same box) $= \frac{1}{365}$

   b. P(Person B puts X on different box) $= 1 - \frac{1}{365} = \frac{364}{365}$

2. a. two

   b. P(Person C has same birthday as Person A or B) $= \frac{2}{365}$

   c. P(Person C has different birthday than Persons A and B) $= 1 - \frac{2}{365} = \frac{363}{365}$

   d. P(Person B has different birthday than Person A and Person C has different birthday than both Person A and Person B)

   $= \frac{364}{365} \cdot \frac{363}{365} = \frac{132{,}132}{133{,}225} \approx 0.9918 = 99.18\%$

   e. P(at least one shared birthday among 3 people) $= 1 - \frac{132{,}132}{133{,}225} = \frac{1{,}093}{133{,}225} \approx 0.0082 = 0.82\%$

3. a. P(Person D has same birthday as Person A, B, or C) $= \frac{3}{365}$

   b. P(Person D has different birthday than Persons A, B, and C) $= 1 - \frac{3}{365} = \frac{362}{365}$

   c. P(four people have four distinct birthdays) $= \left(\frac{364}{365} \cdot \frac{363}{365}\right) \cdot \frac{362}{365} = \frac{47{,}831{,}784}{48{,}627{,}125}$

   $\approx 0.9836 = 98.36\%$

   d. P(at least one shared birthday among 4 people) $= 1 - \frac{47{,}831{,}784}{48{,}627{,}125} = \frac{795{,}341}{48{,}627{,}125}$

   $\approx 0.0164 = 1.64\%$

4. P(k people have k distinct birthdays) $= \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \ldots \cdot \frac{365 - (k - 1)}{365}$

$$= \frac{364 \cdot 363 \cdot 362 \cdot \ldots \cdot (365 - k + 1)}{365^{k-1}} = \frac{364! \div (365 - k)!}{365^{k-1}}$$

$$= \frac{365! \div (365 - k)!}{365^{k}} = \frac{365!}{365^{k}(365 - k)!}$$

(Any of these expressions is an acceptable answer.)

5. P(2 people have 2 distinct birthdays) $= \frac{364}{365} \approx 0.9973 = 99.73\%$

6. Sample answers:
   P(10 people have 10 distinct birthdays) $\approx 0.8831 = 88.31\%$
   P(15 people have 15 distinct birthdays) $\approx 0.7471 = 74.71\%$
   P(20 people have 20 distinct birthdays) $\approx 0.5886 = 58.86\%$

7. P(23 people have 23 distinct birthdays) ≈ 0.49
   In a group of 23 random people, it is more likely that two people share a birthday than that all 23 birthdays are distninct.